Measurement and meaningfulness in Preference Modeling

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Chapter 2

Outline

Numbers

- Today, mathematics are used in all the fields of human activity, not only as a tool to make calculations, but also in the education, the methodology and everyone's way of thinking: we all reason in terms of measures, percentages, ratios, logical deductions, statistics,...
- The constitution of a system of numbers is already a mathematical theory, with many rules, conventions or axioms. These rules can be different depending on what these numbers represent.

Numbers and measurement

- Numbers are used to measure many things: speed, age, density, score, social impact, economic index, probability, possibility, credibility, preference intensity, latitude, date, earthquake intensity, popularity, prediction, ...
- Manipulating "numbers" in social sciences, as most of the decision aiding tools try to do, raises the question of measuring human or social characteristics, such as satisfaction, risk aversion, preference, group cohesion, etc.
- It seems clear that the numbers representing measures cannot always be treated in the same way because the underlying information can be completely different from one context to another.

Introduction to measurement theory

- Measurement: assignment of numbers to attributes of the natural world; It is central to all scientific inference
- Measurement theory concerns the relationship between measurements and reality
- Its goal is ensuring that inferences about measurements reflect the underlying reality we intend to represent.

Introduction to measurement theory

- The key principle of measurement theory is that theoretical context, the rationale for collecting measurements, is essential to defining appropriate measurements and interpreting their values
- Theoretical context determines the scale type of measurements and which transformations of those measurements can be made without compromising their meaningfulness.
- Despite this central role, measurement theory is almost unknown by some practitioners of Decision Modeling, and its principles are frequently violated.

Example (The race)

The arrival order in a race is the following: Alfred, Bill, Carl, David, Ernest, Franz, Gilles, Henry, Isidore and John.

- Team *a*: Alfred, David, Franz and John
- Team *b*: Bill, Carl, Ernest, Gilles, Henry, Isidore
- The duration of the race has been registered, in seconds, yielding for each runner, giving the numbers in this Table:

A	B	C	D	E	F	G	H	I	J
43.5	43.7	44.2	45	47	48	52	52.1	52.5	55

- The purpose is to compare these two teams and, if possible, to decide which team is the best.

Example (The race (times in seconds))

Team a=$\{A; D; F; J\}$ and Team b=$\{B; C; E; G; H; I\}$

A	B	C	D	E	F	G	H	I	J
43.5	43.7	44.2	45	47	48	52	52.1	52.5	55

On the basis of these numbers, are the following assertions valid (true or false)?

1. The mean time of Team b is higher than the mean time of Team a;
2. The second best (lowest) time in Team b is lower than the second best time in Team a;
3. The difference between the best and the worst times in Team a is equal to 11.5.

Example (The race (times in seconds))

Team a=$\{A; D; F; J\}$ and Team b=$\{B; C; E; G; H; I\}$

A	B	C	D	E	F	G	H	I	J
43.5	43.7	44.2	45	47	48	52	52.1	52.5	55

- Convert all the times into minutes
- After this changement, the previous assertions remain true or false?

Example (The weather (Celsius degrees))

Temperatures were measured at noon in two European countries, during respectively 10 and 8 consecutive days. The results, in Celsius degrees, are :

	1	2	3	4	5	6	7	8	9	10
a	20	16	15	14	14	15	13	15	16	18
b	14	12	13	15	14	13	15	16	-	-

- On the basis of these observations, how could we help a tourist choose a country for his holidays?

Example (The weather (Celsius degrees))

	1	2	3	4	5	6	7	8	9	10
a	20	16	15	14	14	15	13	15	16	18
b	14	12	13	15	14	13	15	16	-	-

On the basis of these numbers, are the following assertions valid (true or false)?

1. The mean temperature in country a is higher than the mean temperature in country b;
2. The highest temperature in country a is more than 1.5 times the lowest temperature in country b;
3. The sum of the three highest temperatures in country a is larger than the sum of the four lowest temperatures in country b;

Example (The weather (Celsius degrees))

	1	2	3	4	5	6	7	8	9	10
a	20	16	15	14	14	15	13	15	16	18
b	14	12	13	15	14	13	15	16	-	-

- Convert into Fahrenheit degrees the number (in order to limit the number of decimals, we simply multiply by 1.8 and added 32)
- After this changement, the previous assertions remain true or false?

Example (The weather)

- For example, let us verify that the following assertion is not invariant for the transformation $\alpha x + \beta$: "this temperature is the double of that one".
- Numerically, this assertion can be written

$$x_1 = 2x_2$$

.

- As this equality does not imply that:

$$\alpha x_1 + \beta = 2(\alpha x_2 + \beta), \ \forall \alpha > 0, \ \forall \beta$$

the veracity (resp. falsity) of the assertion can change for an admissible change of scale.

Example (The weather)

- On the contrary, the assertion "this difference of temperature is the double of that one" remains true (resp. false) when an admissible change of scale is applied.
- Indeed,

$$x_1 - x_2 = 2(x_3 - x_4)$$

implies, $\forall \alpha > 0,\ \forall \beta$,

$$(\alpha x_1 + \beta) - (\alpha x_2 + \beta) = 2[(\alpha x_3 + \beta) - (\alpha x_4 + \beta)]$$

Definitions

- "Evaluating" an object consists in associating an element of a numerical scale (a subset of real numbers) to it, according to some conventions as, for example, the choice of a measurement instrument
- The evaluation of an object along a numerical scale is supposed to characterize or to represent a particular information about certain aspects of this object (weight, temperature, age, number of votes, development of a country, air quality, hospitals rankings, hotels ranking, etc.)
- Changing the conventions leads to changing the evaluations of the objects.

Definitions

- Different numerical scales are considered as being “equivalent” if they support (represent) the same information about the considered objects: we will call them “info-equivalent”.
- “Admissible transformations” means ”transformations into info-equivalent numerical scales”.

Types of scale: the ordinal scale

- A scale is *ordinal* if its admissible transformations are all strictly increasing transformations;

Types of scale: the interval scale

- It is *an interval scale* if its admissible transformations are all positive affine transformations of the form

$$\phi(x) = \alpha\ x + \beta \text{ (with } \alpha > 0)$$

In this case, the scale is univocally determined by the choice of an origin and a unit

Types of scale: the ratio scale

- It is a ratio scale if its admissible transformations are the positive homothetic transformations of the form

$$\phi(x) = \alpha\ x\ (\text{with } \alpha > 0)$$

  In this case, the scale is univocally determined by the choice of a unit, the origin being "naturally fixed"

Types of scale: the absolute scale

- The absolute scale does not accept any admissible transformation (except the identity)
- Ex: a counting or a probability scale

Remark

In many cases, it is not possible to characterize the transformations between info-equivalent numerical scales in an analytical way

Meaningful assertion

- In classical measurement theory, an assertion is declared to be meaningful if its truth value is unchanged when admissible transformations are applied to the scales used in the assertion
- More generally (when the admissible transformations are not identifiable), we will say that an assertion is meaningful if its truth value is unchanged when the numerical scales used in the assertion are replaced by info-equivalent scales

Idea of the numerical representation

We try to construct a binary relation $\succsim$ on X such that there exists a numerical function $f : X \longrightarrow \mathbb{R}$ satisfying the property:

$$x \succsim y \Longleftrightarrow f(x) \geq f(y)$$

In general, $\succsim$ is assumed to be a preorder.

- $x \succsim y$ means x is at least as good as y
- $\succ$ is the asymmetric part of $\succsim$
- $\sim$ is the symmetric part of $\succsim$

Theorem (Cantor, 1895)

Let be X a countable set (finite or infinite countable). Let be $\succsim$ a binary relation on X.

$$\left[\exists\, f : X \longrightarrow \mathbb{R} \ s.t.\ \forall x, y \in X,\ x \succsim y \Longleftrightarrow f(x) \geq f(y)\right]$$

$$\Updownarrow$$

$$\succsim \ \textit{is a complete preorder on } X$$

Proposition

Let be $\succsim$ *a preorder (complete) on* X *representable by a function* $f : X \longrightarrow \mathbb{R}$ *i.e.* $\forall x, y \in X, x \succsim y \iff f(x) \geq f(y)$

The following two properties are equivalent:

(i) $v : X \longrightarrow \mathbb{R}$ *is a function representing* $\succsim$

(ii) *There exists a strictly increasing function* $\varphi : f(X) \longrightarrow \mathbb{R}$ *such that* $v = \varphi \circ f$

Remark

f is an ordinal scale

Types of scale: the ordinal scale

- A scale is *ordinal* if its admissible transformations are all strictly increasing transformations;
- Let be $\succsim$ a binary relation on X such that

$$\forall a, b \in X, \begin{cases} a \succ b \Leftrightarrow f(a) > f(b) & a \text{ “is preferred to” } b \\ a \sim b \Leftrightarrow f(a) = f(b) & a \text{ “is indifferent to” } b \end{cases}$$

  - $\succ$ and $\sim$ are invariant for any strictly increasing transformation of the scale of X (leading to an info-equivalent scale).
  - The representation of $\succ$ and $\sim$ leads to an ordinal scale.
  - Every assertion based on these relations can thus be considered as “meaningful”.

Types of scale: the interval scale

- It is *an interval scale* if its admissible transformations are all positive affine transformations of the form $\phi(x) = \alpha\, x + \beta$ (with $\alpha > 0$)
- Let be $\succsim$ a binary relation on X such that

$$\forall a, b, c, d \in X, \begin{cases} a \succ b \Leftrightarrow f(a) > f(b) & a \text{ "is preferred to" } b \\ a \sim b \Leftrightarrow f(a) = f(b) & a \text{ "is indifferent to" } b \\ (a, b) \succ^* (c, d) \Leftrightarrow f(a) - f(b) > f(c) - f(d) & \\ (a, b) \sim^* (c, d) \Leftrightarrow f(a) - f(b) = f(c) - f(d) & \end{cases}$$

  - $(a, b) \succ^* (c, d)$ means "the preference of a over b is stronger than that of c over d" and $(a, b) \sim^* (c, d)$ means "the preference of a over b is as strong as that of c over d".
  - The representation of $\succ$, $\succ^*$, $\sim$ and $\sim^*$ leads to an interval scale.
  - $\succ$, $\succ^*$, $\sim$ and $\sim^*$ are invariant for any positive affine transformation of X (leading to an info-equivalent scale), so that assertions solely based on them are meaningful.

Types of scale: the ratio scale

- It is a ratio scale if its admissible transformations are the positive homothetic transformations of the form

$$\phi(x) = \alpha\ x\ (\text{with}\ \alpha > 0)$$

In this case, the scale is univocally determined by the choice of a unit, the origin being "naturally fixed"

Be careful

- As we see, depending on the scale type (i.e. depending on the information supported by the scale), some caution is necessary in the manipulation and the interpretation of the numbers if we want to obtain meaningful conclusions based on these numbers.
- A conclusion that is true using a given scale but that is meaningless (not meaningful) for this type of scale is completely dependent of the particular scale which is considered, has no character of generality and is thus, probably, of very limited interest.
- **It can even be dangerous because of the tendency of humans to generalize ideas without sufficient precautions.**
- The analysis of scale types allows to detect manipulations (mathematical operations) which can lead to meaningless conclusions

Be careful

- Scale types are not "naturally given" in decision aiding, even for physical measures, and that every use of numbers must be accompanied by some precisions on the information they are supposed to support
- Meaningfulness theory is an important tool for the analysts in order to avoid the development of completely arbitrary decision aiding procedures
- In evaluation and decision problems, the nature of the numbers used is partially in the hands of the analyst: it mainly depends on the purpose of the decision aiding process and on the future steps of the process (is it really useful to build a ratio scale if the next step only exploits the ordinal properties of the numbers?).
- The role of the analyst is to be sure that all the operations are compatible with his choice, from the assessment of the numbers to their interpretation, including the mathematical manipulations of these numbers.

Be careful

- Even if the numbers are useful, their presence in a "model" does not guarantee that it is a formal model. In a sense, the ease of use of the numbers may be a pitfall since it can lead to instrumental bias.
- Another confusion is often made between the term "qualitative" and the absence of numerical information.
  - The color of an object is typically qualitative but can be represented by a number (the wave length).
  - On the contrary, the expression "a small number of students" does not contain any number but is certainly not qualitative. It represents a quantity.

Reference

Denis Bouyssou, Thierry Marchant, Marc Pirlot, Alexis Tsoukiàs, and Philippe Vincke.
Evaluation and decision models with multiple criteria: Stepping stones for the analyst.
International Series in Operations Research and Management Science, Volume 86.
Boston, 1st edition, 2006.